Mere Addition is equivalent to avoiding the Sadistic Conclusion in all plausible variable-population social orderings

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Abstract

Economic policy evaluations require social welfare functions for variable-size populations. Two important axioms in the population ethics literature are Mere Addition and avoidance of the Sadistic Conclusion, both of which focus on the sign of lifetime utility. The population ethics literature treats these axioms as closely related but distinct: one influential review calls avoidance of the Sadistic Conclusion "less controversial." Here, we provide weak, uncontroversial sufficient conditions for these two principles to be equivalent. Related results exist in prior literature, but these include only same-number utilitarian orderings and therefore exclude recent and theoretically important rank-dependent social evaluations that we include. [100 words]

Keywords: social welfare, population ethics, mere addition, sadistic conclusion, utilitarianism, rank-discounted utilitarianism

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1 Introduction

Population ethics studies the axiomatic properties of social orderings of welfare vectors with different population sizes. Two well-studied axioms in population ethics hinge on the sign of lifetime utilities. Mere Addition, introduced by Parfit (1984), holds that adding a positive-utility life to any population does not result in a worse population. The Sadistic Conclusion, introduced by Arrhenius (2000), holds that adding negative lives can be strictly preferred to adding positive lives. The literature typically describes the Sadistic Conclusion as ethically worse than denying Mere Addition. Arrhenius (n.d.) calls the requirement to avoid the Sadistic Conclusion a "less controversial assumption" than Mere Addition (p. 297).

Arrhenius (n.d.) notes that social orderings that violate Mere Addition "tend to imply" the Sadistic Conclusion (p. 94). Greaves (2017) calls these axioms "related." Bossert (2017) especially highlights the normative link but less generally than we do. We show the link is stronger than tendency. Under a set of basic plausibility axioms not violated by any social ordering defended in the economics literature, these two conditions are equivalent. In their theorem 5.4, Blackorby, Bossert, and Donaldson (2005) state a related result [1] that only applies to social orderings that are same-number generalized utilitarian (e.g., not rank-dependent evaluations, an important recent innovation by Asheim and Zuber (2014)).

2 Setting and basic axioms

Our notation follows Blackorby, Bossert, and Donaldson (2005). $\mathbb{Z}$ is the integers, $\mathbb{R}$ is the real numbers, $\mathbb{R}_{++}$ and $\mathbb{R}_{+}$ are the positive and nonnegative real numbers, respectively, and similarly for $--$, $-$, and $\mathbb{Z}$.

Populations $\mathbf{u}, \mathbf{v}$ are finite-length vectors of real numbers, where the ith position in the vector u_i is the lifetime utility of person i. Utilities are normalized so that $u_i = 0$ is a neutral life (as good as a life with no experiences) for person i. Following Asheim and Zuber (2014), an index enclosed in square brackets indicates rank from worst-off.

The size of $\mathbf{u}$ is $n(\mathbf{u}) \in \mathbb{Z}_{++}$, so $\mathbf{u} \in \mathbb{R}^{n(\mathbf{u})}$. For same-sized populations, $\mathbf{u} \geq \mathbf{v}$ means $u_i \geq v_i$ for all i; $\mathbf{u} > \mathbf{v}$ means $\mathbf{u} \geq \mathbf{v}$ and $u_i \neq v_i$ for some i; $\mathbf{u} \gg \mathbf{v}$ means $u_i > v_i$ for all i.

[1] Their theorem does not directly claim Mere Addition and avoiding the Sadistic Conclusion are identical but rather uses the Sadistic Conclusion in a Repugnant Conclusion impossibility theorem. So does Arrhenius (2000) in his introductory use of the Sadistic Conclusion.

$\mathbf{1}_n$ is an n-dimensional vector of ones, so $\xi\mathbf{1}_n$ is n people with utility ξ each. $\cup$ notation combines populations, so $n(\mathbf{u}\cup\mathbf{v}) = n(\mathbf{u}) + n(\mathbf{v})$.

The set of all possible populations is $\Omega = \bigcup_{n\in\mathbb{Z}_{++}} \mathbb{R}^n$. Ω_{--} and Ω_{++} are the subsets of Ω with all negative or positive lives, respectively. This paper describes $\succsim$, a social welfare relation on Ω. The asymmetric and symmetric parts of $\succsim$ are $\succ$ and $\sim$, respectively.

We adopt three uncontroversial basic axioms from Blackorby and Donaldson (1984) as minimal criteria for plausibility.

Axiom (Social Order)**.** *The relation $\succsim$ is complete, transitive, and reflexive on Ω.*

Axiom (Continuity)**.** *For all $n, m \in \mathbb{Z}_{++}$ and all $\mathbf{u} \in \mathbb{R}^n$, the sets $\{\mathbf{v}\in\mathbb{R}^m : \mathbf{v}\succsim\mathbf{u}\}$ and $\{\mathbf{v}\in\mathbb{R}^m : \mathbf{v}\precsim\mathbf{u}\}$ are closed in $\mathbb{R}^m$.*

Axiom (Same-number Pareto)**.** *For all $\mathbf{u},\mathbf{v}\in\Omega$ such that $n(\mathbf{u}) = n(\mathbf{v})$, if $\mathbf{u} > \mathbf{v}$ then $\mathbf{u}\succ\mathbf{v}$.*

3 Mere Addition and the Sadistic Conclusion

Axiom (Mere Addition)**.** *For all $\mathbf{u}\in\Omega$ and all $\mathbf{v}\in\Omega_{++}$, $\mathbf{u}\cup\mathbf{v}\succsim\mathbf{u}$.*

Axiom (Avoidance of the Sadistic Conclusion)**.** *For all $\mathbf{u}\in\Omega$, all $\mathbf{v}\in\Omega_{++}$, and all $\mathbf{w}\in\Omega_{--}$, $\mathbf{u}\cup\mathbf{v}\succsim\mathbf{u}\cup\mathbf{w}$.*

Total Utilitarianism ($W(\mathbf{u}) = \sum_i u_i$) and Total Prioritarianism ($W(\mathbf{u}) = \sum_i g(u_i)$ for increasing, concave g such that $g(0) = 0$) satisfy Mere Addition and avoid the Sadistic Conclusion. Average Utilitarianism ($W(\mathbf{u}) = \frac{1}{n(\mathbf{u})}\sum_i u_i$), Average Prioritarianism ($W(\mathbf{u}) = \frac{1}{n(\mathbf{u})}\sum_i g(u_i)$), Number-Dampened Utilitarianism ($W(\mathbf{u}) = \left(\frac{1}{n(\mathbf{u})}\sum_i u_i\right) n(\mathbf{u})^\alpha$, where $0<\alpha<1$, yielding a positive, increasing, concave transformation of population size; abbreviated NDGU), Critical-Level Generalized Utilitarianism ($W(\mathbf{u}) = \sum_i (g(u_i) - g(c))$ for positive c; abbreviated CLGU), and Rank-Dependent Generalized Utilitarianism ($W(\mathbf{u}) = \sum_{r=1}^{n(\mathbf{u})}\beta^r g\left(u_{[r]}\right)$, for $0<\beta<1$; abbreviated RDGU) all violate Mere Addition and imply the Sadistic Conclusion.

RDGU is notable because Asheim and Zuber (2014) emphasize — in presenting RDGU to the literature — that it escapes a conclusion called the Very Sadistic Conclusion. However, as Pivato (2019) shows, RDGU implies the Sadistic Conclusion, the principle we consider here.

Maximin violates Mere Addition but avoids the Sadistic Conclusion. Social ordering according to size ($W(\mathbf{u}) = n(\mathbf{u})$) satisfies Mere Addition but does not avoid the Sadistic Conclusion.

4 Result

Axiom (Consistent Expansion)**.** *For all* $\mathbf{u} \in \Omega$, $v \in \mathbb{R}$, *and* $n \in \mathbb{Z}_{++}$, *if* $\mathbf{u} \succ \mathbf{u} \cup v\mathbf{1}_n$ *then there exists* $m > n$ *such that* $\mathbf{u} \cup v\mathbf{1}_n \succ \mathbf{u} \cup v\mathbf{1}_m$.

Consistent Expansion holds that if adding n v-lives to $\mathbf{u}$ makes a worse combined population, then adding some further number of v-lives makes an even worse population. [2] Consistent Expansion is satisfied by Total and Average Utilitarianism and Prioritarianism, Critical-Level Leximin, NDGU, CLGU, and RDGU. Consistent Expansion is failed by Maximin but is trivially satisfied by ordering according to size, for which the antecedent condition is never met.

Proposition 1. *In the context of the basic axioms and Consistent Expansion, Avoidance of the Sadistic Conclusion implies Mere Addition.*

Proof. By contrapositive. Assuming Mere Addition is violated, there exist $\mathbf{u} \in \Omega$ and $v \in \mathbb{R}_{++}$ such that $\mathbf{u} \succ \mathbf{u} \cup v\mathbf{1}_1$. By Pareto, $\mathbf{u} \cup v\mathbf{1}_1 \succ \mathbf{u} \cup 0\mathbf{1}_1$, so $\mathbf{u} \succ \mathbf{u} \cup 0\mathbf{1}_1$ by Transitivity. Next, by Consistent Expansion, there exists $m > 1$ such that $\mathbf{u} \cup 0\mathbf{1}_1 \succ \mathbf{u} \cup 0\mathbf{1}_m$. Thus, by continuity, there is an open ball around 0 of diameter δ_1 such that if $|\varepsilon_1| < \delta_1$ then $\mathbf{u} \cup \varepsilon_1\mathbf{1}_1 \succ \mathbf{u} \cup 0\mathbf{1}_m$. Similarly, there is an open ball around 0 of diameter δ_2 such that if $|\varepsilon_2| < \delta_2$ then $\mathbf{u} \cup \varepsilon_1\mathbf{1}_1 \succ \mathbf{u} \cup \varepsilon_2\mathbf{1}_m$. Choosing $\varepsilon_1 < 0$ and $\varepsilon_2 > 0$ within these open balls gives an instance of the Sadistic Conclusion. □

Mere Addition is related to another axiom about the sign of utility:

Axiom (Negative Expansion)**.** *For all* $\mathbf{u} \in \Omega$ *and all* $\mathbf{v} \in \Omega_{--}$, $\mathbf{u} \succsim \mathbf{u} \cup \mathbf{v}$.

Negative Expansion says that adding a set of negative lives to a population makes the combined population worse. Total Utilitarianism and Prioritarianism, CLGU, and RDGU all satisfy Negative Expansion. But Average Utilitarianism and NDGU violate Negative Expansion. As Pivato (2019) documents, given Negative Expansion, Mere Addition implies Avoidance of the Sadistic Conclusion.

Proposition 2. *In the context of Social Order and Negative Expansion, Mere Addition implies Avoidance of the Sadistic Conclusion.*

[2] Arrhenius's (2000) "Addition Principle" is related; it is implied for some m by the conjunction of Consistent Expansion, Pareto, and Transitivity.

Proof. Let $\mathbf{u} \in \Omega$, $\mathbf{v} \in \Omega_{++}$, and $\mathbf{w} \in \Omega_{--}$. By Mere Addition, $\mathbf{u} \cup \mathbf{v} \succsim \mathbf{u}$, and by Negative Expansion, $\mathbf{u} \succsim \mathbf{u} \cup \mathbf{w}$. Hence, by transitivity, $\mathbf{u} \cup \mathbf{v} \succsim \mathbf{u} \cup \mathbf{w}$, avoiding the Sadistic Conclusion. □

Having ruled out, in Proposition 1, cases that avoid the Sadistic Conclusion without implying Mere Addition, we can conclude that these two axioms are identical if we can further know that all approaches that satisfy Mere Addition also avoid the Sadistic Conclusion. We know this trivially of the many approaches, such as Average Utilitarianism, that reject Mere Addition. By Pivato's logic in Proposition 2, we know this also of any approach that satisfies both Mere Addition and Negative Expansion. The remaining category of approaches would be those that satisfy Mere Addition but reject Negative Expansion. But none of these are proposed or defended in the literature. They are logically possible: CLGU with a negative critical level ($W(\mathbf{u}) = \sum_i \left(g(u_i) - g(d)\right), d < 0$), for example, satisfies Mere Addition and rejects Negative Expansion; it also implies the Sadistic Conclusion by strictly preferring the addition of some negative lives to any population. But this example, and any such approach, is normatively implausible.

5 Discussion

Ng's (1989) classic impossibility theorem shows that no social ordering can:

- satisfy Mere Addition,
- satisfy a normatively unobjectionable principle called Non-Antiegalitarianism, and
- avoid Parfit's (1984) Repugnant Conclusion.

Because the economics literature has never entertained violating Non-Antiegalitarianism, in practice one must reject Mere Addition or accept Parfit's Repugnant Conclusion. One implication of our result is that, in the context of the specified axioms, satisfying Mere Addition can be replaced with avoiding the Sadistic Conclusion.

Although one response to our result is to deny our axioms as minimal requirements for plausibility, no candidate approach in the economics literature does so. This may increase the pressure for one to accept Parfit's Repugnant Conclusion. Alternatively, one may accept the Sadistic Conclusion. Bossert (2017), out of commitment to existence independence and avoiding the Repugnant Conclusion, does so. Bossert's rationale coheres with this paper's link between Mere Addition and the Sadistic Conclusion: "from the viewpoint of a critical-level utilitarian who supports the use of a positive critical level, the competing additions of individuals to a given population are not very attractive to begin with."

A broader implication of our paper is towards unifying and organizing the many axioms and principles in population ethics. In the three decades since Ng's proof, population ethicists have focused on developing richer versions of the impossibility theorem with complex axioms (Blackorby, Bossert, Donaldson, and Fleurbaey, 1998; Arrhenius, 2000, n.d.). Under basic plausibility axioms for any candidate social ordering, however, this complexity can be simplified in a way that clarifies the basic theoretical trade-offs.

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